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THE LAWS OF INCREASING AND DIMINISHING RETURNS

[THE following article, published in the ECONOMIC JOURNAL 1911, formed (the greater part of) an introduction to a series of articles entitled *Contributions to the Theory of Railway Rates;* which was discontinued after 1913. This discussion of Increasing and Diminishing Returns which initiated the series is almost as applicable to a regime of Competition as to one of Monopoly. But its original use as an introduction to an essay on a certain kind of monopoly is occasionally discernible.

There are here distinguished two definitions of increasing (and of diminishing) returns, which are respectively appropriate according as the magnitude of which increase (or diminution), is predicated is marginal or total. It is proposed to remove an ambiguity, the existence of which in so familiar a subject might have been deemed impossible, but that a similar confusion occurs in the matter of "sacrifice" incurred through taxation as pointed out below II. 115, and perhaps even (as there suggested) in a still higher sphere. The definition based on marginal production is here distinguished as *primary*, preferred as more directly related to theory of *maxima*. The use of the *secondary* definition is shown to present difficulties in the important case of plural factors of production.

The *conceptions* having been made clear, some of the *propositions* of which they form the terms are restated. The advantages of Production-on-a-large-scale and of Division-of-labour are classified.

There follows an inquiry into the meaning and properties of Joint Cost, and other cognate conceptions. In the course of the inquiry there comes into view that case of which the title to Joint Production has been disputed between Professor Pigou and other high authorities (see Index s.v. *Joint Production*).]

Increasing Returns and Joint Cost admittedly play a great part in economics; but what that part is has been questioned.

1. *Law meaning concept.*—A first difficulty is occasioned by a certain ambiguity in the use of the word " law " in such phrases as the " law of increasing [or diminishing] returns." The word " law " in this connection is used sometimes in the narrow sense of a quantitative relation; sometimes in the larger sense connecting that conception with some other attribute. Thus the " law of diminishing returns " may stand either for the conception of decrease in the rate at which production increases, or for a proposition connecting that conception with the increase in the number of producers under certain circumstances. Accordingly Mr. Flux well distinguishes the " definition or statement of the law " from the " assertion of its applicability." [1] So Mr. Maurice Clark, in his philosophical study on *local freight discriminations*, appears to treat the " law of joint cost " as equivalent to the " term ' joint cost.' " [2]

This sort of ambiguity is not unknown in other sciences. Thus the " law of error " is sometimes used to denote the relation between frequency and deviation which is expressed by a certain well-known curve (or *function*), and sometimes for the proposition that the said quantitative relation tends to be realised approximately in certain circumstances of general occurrence. Even in physical science such a phrase as " the law of the inverse square " is not improperly, I think, sometimes used to denote simply a certain relation between the magnitude of a force and the distance at which it operates, a conception which may be predicated of different forces—now the attraction of gravitation, and now the repulsion of electricity.

We may begin by interpreting the laws in question in the narrower sense. Let it not be objected that this is a matter of verbal definition. For, as Sidgwick reminds us, some of the most important inquiries have taken the form of a search for definitions.[3] More reassuring to some may be the reflection that even in modern physical science half the battle often consists in obtaining what Whewell well described as clear and appropriate conceptions, " ideas distinct and appropriate to the facts." [4]

2. *Provisional definition.*—The definition of the law as a *term* may be gathered from an authoritative statement of the law as a

[1] Palgrave's *Dictionary*, Article on Law, p. 583.

[2] Pp. 28, 29, 30.

[3] *Political Economy*, Book I. ch. ii. § 1, suggesting the application of this Platonic method to economic investigations; *cf.* Book II. ch. iv. *note* (ed. 3).

[4] *History of the Inductive Sciences*, Book I. ch. iii. § 2 *et passim*. Mill, while refusing to " Colligation " the title of Induction, does not deny its supreme importance.—*Logic*, Book II. § 4.

proposition. The essential attribute is presented in the following passage—which want of space compels me to separate from the explanation and limitations in the context—from Dr. Marshall's statement of the law of diminishing return with respect to agriculture :—" Our law states that sooner or later . . . a point will be reached after which all further doses will obtain a less proportionate return than the preceding doses." [1] So with respect to manufactures, Dr. Marshall says :—" If a manufacturer has, say, three planing machines, . . . after they are once well employed, every successive application of effort to them brings him a diminishing return." [2] So Mr. Flux discriminates between Increasing or Diminishing Return by the change in " marginal expenses per unit." [3] We are countenanced, I think, by good authority in adopting the following provisional definition of the terms. When on the application of two successive equal doses of productive power, the increment of product due to the first dose is less than the additional increment due to the second, the law of increasing returns is said to act; and conversely it is a case of diminishing returns when the increment due to the first dose is greater than the increment due to the second.

The attribute which I regard as essential may be illustrated by a numerical example. In the accompanying table the first two columns are borrowed from an example given by Professor Carver.[4]

TABLE I.

Days' labour of man with team and tools.	Total crop in bushels.	Increments due to successive doses.
2	0	0
5	50	50
10	150	100
15	270	120
20	380	110
25	450	70
30	510	60
35	560	50
40	600	40
45	630	30
50	650	20

[1] *Principles of Economics*, sixth edition, p. 153.

[2] *Op. cit.* p. 168.

[3] *Economic Principles*, p. 47.

[4] See Prof. Carver's *Distribution of Wealth*, ch. ii. p. 58, and compare his article in the ECONOMIC JOURNAL, Vol. XVIII. A part of Prof. Carver's third column, not shown in my Table I., is reproduced (with some additional matter) in my Table II. I have also taken the liberty of substituting in his first column for his figure 1 the figure 2.

I have added the third column to illustrate the distinction between Increasing and Diminishing Returns according to the definition here provisionally adopted. The figure in the third column distinguished by heavy type, viz. 120, marks the point up to which returns are increasing and after which they become diminishing.

If we plot a set of figures like those above given and exhibit the relation between the figures in the first and those in the second column in the form of a curve, it will be found that the character of the law (whether "increasing" or "diminishing") depends on the character of the curve in respect of concavity or convexity.*

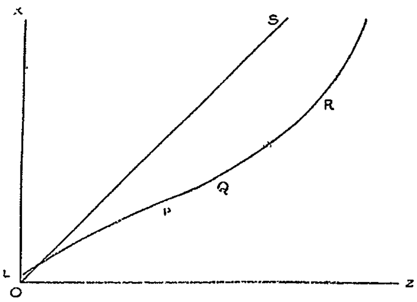

FIG. 1 A.

In Fig. 1 A the vertical axis OX is taken to represent outlay; each quarter of an inch on this ordinate denoting a "dose" of five days' labour. The corresponding returns are represented by the axis OZ; each quarter of an inch on this abscissa denoting fifty bushels. It will be observed that the curve is concave (with respect to the horizontal) up to the third dose, the point P, while Increasing Returns acts; and becomes convex when Diminishing Returns sets in. In Fig. 1 B representing the same data, with the directions of the co-ordinates interchanged, Increasing and Diminishing Returns correspond respectively to the con-

* If z denotes the (amount of) product and x the (amount of) factor used in the production; the curve in Fig. 1 A will be convex, and the curve in Fig. 1 B concave (towards the abscissa), when $\frac{d^2x}{dz^2} > 0$, or $\frac{d^2z}{dx^2} < 0$ (which conditions come to the same, since $\frac{dz}{dx}$ is supposed positive). For

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\frac{dz}{dx} = \frac{d}{dx}1\Big/\frac{dx}{d\cdot} = \left(\frac{d^2x}{dz^2}\cdot\frac{dz}{dx}\right)\Big/\left(\frac{dx}{dz}\right)^2.$$

vexity and concavity of the curve with respect to the horizontal axis.[1]

3. *Doses of various size.*—In order to make our definition precise it is often necessary to specify the *magnitude* of the doses successively applied. Otherwise it may happen that *both* Increasing and Diminishing Returns may truly be predicted of the same circumstances.[2] This is a paradox familar to those who are conversant with the application of the differential calculus. It depends on the circumstance that the orders of magnitude which may be neglected are different according to the different purposes

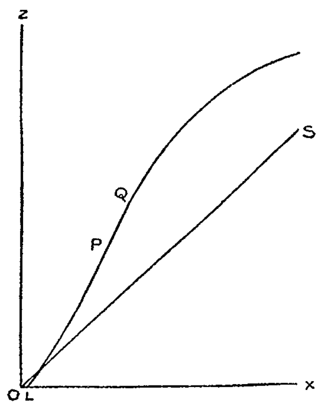

Fig. 1 B.

contemplated. It is thus—to use Clerk-Maxwell's illustration [3]—that the heterogeneities in the structure of a mound of gravel, which are negligible from the point of view of the railway contractor, may be all-important to the worm. For a like reason the surface of a mountain at any assigned point, say that which is exactly underneath the centre of gravity of an ascending mountaineer, may appear to him, while he surmounts the rough surface with long strides, to be shaped like a dumpling, concave with respect to the plane of the horizon; but to the beetle creeping up a cup-shaped cavity, convex.

[1] Of the two nomenclatures, here as usual treated as equivalent, namely increasing [or diminishing] return and diminishing [or increasing] cost, the former seems more appropriate to construction B, the latter to construction A.

[2] *Cf.* Marshall, *Principles*, ed. v. p. 159 and context.

[3] In a passage quoted II. 389.

Similarly in the example above given, if we use doses each consisting of *twenty-five* days' labour, the character of increasing return will no longer be presented. For the return due to the first dose will now be 450; the return due to the second dose 200. Even if we use doses each consisting of *fifteen* days' labour we do not find increasing returns; the returns to the successive doses being 270, 240, 120. So when the passenger traffic on a railway is increased, for small additions requiring only additional carriages on the trains already running, the case may be one of diminishing cost, increasing return; but for large additions requiring additional trains on an already crowded track, the case may be one of increasing cost, diminishing return. Yet again, when the increase of traffic is such as to call for a new track, a change of this magnitude might well present the law of increasing return. In such cases, then, in order that the predication of either law of return should be significant, it is necessary that the size of the dose should be explicitly assigned, if not implied by the context.

4. *General definition.*—So far we have tacitly supposed the two successive doses to be equal in magnitude. But now, removing this restriction, we may define that when on the application of two (not in general equal) doses of productive power the increment of product due to the two doses has to the increment of product due to the first dose alone a ratio greater than the ratio which the sum of the two doses has to the first dose, Increasing Return acts; [1] and conversely if the former ratio is *less* than the latter, Diminishing Return.[2] For example, in the numerical instance above given we may say that after the stage at which thirty-five days' labour (supplemented by team and tools) have been applied, the law of diminishing return acts, since one dose (of five days, etc.)—added to six such doses—produces 60 bushels, while three added doses produce $60 + 50 + 40 = 150$, and the ratio of 150 to 60 is less than the ratio of 3 (doses) to 1. This definition appears to agree with the expressions employed in

[1] In other words (used by the present writer, ECONOMIC JOURNAL, Vol. XIX. p. 293) "the law of increasing cost or diminishing returns holds good when the ratio of the last increment of cost to the last increment of produce is greater than the ratio of the penultimate increment of cost to the penultimate increment of produce; with a corresponding statement for the law of diminishing cost (increasing returns)."

[2] Let $z = f(x)$. At any point x_1 increasing or diminishing return acts according as

$$\frac{f(x_2) - f(x_0)}{f(x_1) - f(x_0)} > \text{ or } < \frac{x_2 - x_0}{x_1 - x_0};$$

where $x_0 < x_1 < x_2$.

many standard treatises. Thus Professor Nicholson predicates of " increasing returns " that " under certain conditions every additional unit of productive power gives more than proportional returns " ; [1] with a corresponding definition of diminishing returns. So Professor Seager states with respect to diminishing returns in agriculture, that, after a certain point, " applications of labour and capital yield less than proportionate returns in product." [2] Such expressions often leave it doubtful whether they were intended to refer to the general definition which has been here enunciated, or only to the particular, though extensive, species which is now to be distinguished.

5. *Division into two species.*—Our general definition comprehends a particular and limiting case in which the difference of degree almost amounts to a difference of kind. This case is constituted by taking as the first of the two successive doses the whole of the labour-and-capital or productive power measured from zero; the second dose being larger or smaller according to the purpose in hand.[3] Thus understood, the definition comes to this, that the law of increasing return acts when the *average* product per unit of productive power applied increases, with the increase of productive power (by an amount that is of an assigned order of magnitude); and the law of diminishing return, in the converse case. The definition thus presented may be distinguished as *secondary ;* the general definition, exclusive or irrespective of this limiting case, being called *primary.*[4] It should be observed, however, that many high authorities seem to give precedence to that definition which is here described as secondary. Thus Walker makes the distinction between increasing and diminishing returns, with reference to a given tract of land cultivated by ten labourers, to turn upon the question whether or not, if two new labourers are taken on the twelve raise more per man than the ten could do. Similarly Professor Seligman [5] and other eminent American economists.[6] So Professor Carver, with reference to the instance above cited from him, understands that " increasing returns stop and diminishing returns begin at the point where twenty days' labour are expended in the cultivation of the field " [7]—that is at the fourth dose (of five days' labour), not as according to our definition the

[1] *Principles of Political Economy,* Vol. I. p. 172.
[2] *Introduction to Economics,* 1904, p. 114.
[3] Putting $x_0 = 0$ in the formula given in note 2, p. 66.
[4] *Cf.* below, p. 152.
[5] *Shifting and Incidence of Taxation,* quoted by the present writer, *loc. cit.*
[6] *E.g.* Bullock, *Elements of Economics,* p. 76.
[7] *Distribution of Wealth,* p. 58.

third dose—because up to that point the average product per dose (or what comes to the same, " per day's labour ") goes on increasing.

To exhibit the distinction more clearly, I suppose Table I. to be graduated more finely in the neighbourhood of the transition

TABLE II

Days' labour of man with team and tools.	Total crop in bushels.	Increments due to successive doses.	Bushels per day's labour.
...	...	...	...
...	...	...	...
13	220	...	16·92
14	244	24	17·43
15	270	26	18
16	294	24	18·38
17	317	23	18·65
18	339	22	18·83
19	360	21	18·95
20	380	20	19
21	396	16	18·86
...	...	...	...
...	...	...	...

from Increasing to Diminishing Returns; and I add a fourth column [1] corresponding to the second definition. It will be seen that for a considerable tract of values—corresponding to the portion of the curve in Fig. 1 A between the points P and Q—the primary and secondary definitions do not come to the same. The difference in connotation involves a sensible difference in denotation.

The grounds on which precedence is claimed for the primary definition will presently appear.

6. *Significance of price.*—So far we have mostly measured the producing doses and the resulting product in *kind*.[2] But no essential difference in classification is introduced when we take money as the measure; provided that the prices, both of the product and the factor of production, remain constant while the amounts are varied. For the change thus introduced is simply to multiply the axes representing outlay and return, the Ox and Oz of Fig. 1 each by a constant: to change the scale of both

[1] Corresponding to Prof. Carver's *third* column.

[2] *Cf.* Marshall, *Principles*, ed. v. p. 152: " the return to capital and labour of which the law [of Diminishing Return] speaks is measured by the *amount* of the produce raised independently of any change that may meanwhile take place in the *price* of produce."

the abscissa and the ordinate.[1] But such a change does not alter the character of a curve in respect of convexity or concavity. If it was convex or concave at any point before the change, the transformed curve will have the same character at the corresponding point. The character of the return, whether increasing or decreasing, in the primary sense, depends on the material conditions of production, not on the accidents of price. With respect to the distinction in the secondary sense, we may employ a theorem given in my former treatment of the subject,[2] that in such a figure as our 1 A above, if a tangent (not shown in the figure) is drawn from the origin to the curve, the point of contact is the limit at which the returns cease to be increasing in the secondary sense and become decreasing. This relation, too, may be considered as an *invariant*, not varying with a change of scale.

But money can no longer be ignored when we consider price as varying with the amount put on the market by the individual entrepreneur; as it is proper to conceive in a regime of monopoly. We must now distinguish z the amount of product in kind due to the employment of the factor x,[3] and ζ the money-value of that product depending on the law of demand.

7. *Factors and other coefficients.*—In general, we may presume that, as shown above in Fig. 1, to any assigned amount of outlay there corresponds a definite amount of product, and conversely. In this presumption it is taken for granted that the entrepreneur applies his outlay to the best of his ability [4] in order to obtain the greatest possible profit. To that end he may have to adjust any number of variables, such as the time of trains, the place of porters, and so forth. We ought to distinguish this sort of coefficient, which does not enter into the expression for outlay * from factors-of-production usually regarded as, in the phrase of a judicious writer, "factors of expense." [5]

[1] If in Fig. 1 A the ordinate represent not x the amount of a factor, but ξ the money value thereof, the curve will then be a cost-curve of the kind employed by Auspitz and Lieben.

Similarly in Fig. 1 B the abscissa may be taken to represent outlay in money.

[2] *Loc. cit.*, p. 294.

[3] Supposed to be purchasable by the monopolist at a constant price.

[4] *Cf.* Marshall, *Principles*, ed. v. p. 152: ". . . taking farmers as they are with the skill and energy which they actually have." *Cf.* also the passage cited in the sequel (p. 97), with reference to Prof. Carver's views.

* These *gratuitous* coefficients may be identified with the "parameters" u, v, w . . . which Dr. Zotoff in his elaborate note on the Mathematical Theory of Production (Economic Journal, Vol. XXXIII. p. 115) introduced and eliminated.

[5] Johnson (and Huebner), *Railway Traffic and Rates.*

Here might appropriately follow the discussion of plural factors of production. But it is better first, still with special reference to a single simple factor, to advert to the grounds on which different definitions are preferred.

8. *The two species compared.*—Things which are insignificant for the purposes of action and pleasure do not obtain names. What then is the purpose with reference to which the names now in question have been imposed? The essential fact, I submit, is that the attribute designated Diminishing Return is the criterion of a *maximum ;* not only of a quantity such as z, the product considered as a function of x, the factor used, but also of a quantity such as $bz - cx$ (where b and c are constants), denoting the net product.[1]

Moreover, it is to be conceived that an analogous condition is fulfilled by the *gratuitous* coefficients above noticed,[2] though the vocabulary of the economist may fix attention on the paid factors of production. For instance, in the case of a given amount of labour and capital to be applied to an optional amount of land,[3] the condition which must be fulfilled by the law of production in order that the product should be a maximum is the same whether the land is free, or subject to a rent per acre.[4]

How comes it, then, that the *secondary* definition is so largely employed by economists? For one reason, there is often no difference in the denotations corresponding to the different connotations. This occurs when the cost-curve represented in Fig. 1 A is convex (to the abscissa), *ab initio.*[5] This coincidence of fact may explain the frequent use of different definitions by the same writer.[6]

[1] *Cf.* below, p. 74. In the abstract b and c may be not prices, but quantities of some commodity other than money, in particular the commodity produced.

[2] Above, subsection 7.

[3] As in Prof. Carver's instructive example above cited.

[4] Let the product be $z, = f(x, l)$, where x is the amount of working capital, l of land employed; and let the net product be $f(x, l) - c_1x - c_2l$, where c_1, c_2 are constants (*cf.* note 1 above). The criterion of a maximum, namely, that the second term of variation should be thoroughly negative, is the same whether c_2 is zero or not.

[5] *Cf.* below, p. 157.

[6] The coincidence is thus affirmed by one who was among the first to discern the principle of Diminishing Returns—West :—"each additional quantity of work bestowed on agriculture yields an actually diminished return, and, of course, if each additional quantity of work yields an actually diminished return, the whole of the work bestowed on agriculture in the progress of improvement yields an actually diminished proportional return."—*Essay on the Application of Capital to Land*, pp. 6–8, quoted by Prof. Cannan, ECONOMIC JOURNAL, Vol. II. p. 63.

But this identity is not always to be supposed. Rather, the curve in Fig. 1 A is typical of many modern industries in which an initial outlay is required. What is the rôle of the *secondary* definition in such cases? Let us consider this nice question with reference to three kinds of economic regime: (*a*) perfect competition, (*b*) monopoly practised by a perfectly self-interested monopolist, (*c*) monopoly practised, or at least regulated, by the State.

(*a*) In the first case it is proper to suppose a constant price at which each entrepreneur strives to sell that amount of product which brings him in a maximum profit. If in Fig. 1 A the constant price is represented by the inclination (to the abscissa) of a straight line through the origin [1] (the axis *OZ* now representing the money-value of any quantity of product *z*), then the amount produced by an individual whose cost-curve [2] is *OLPQR* will be the abscissa to the point on the curve, which is such that a tangent to the curve at that point is parallel to the line *OS*; provided that the curve is convex (to the abscissa) at that point.[3] Now at first sight it might appear that this condition could be satisfied by the convex portion of the curve in Fig. 1 A, between *P* and *Q*, if the price were suitable. But the condition will be found to imply that the total gain obtained from the production is less than the total loss incurred; which is, normally and in the long run, absurd. Accordingly, we are concerned (in a regime of competition) only with that part of the curve which fulfils the secondary as well as the primary definition, the tract beyond *Q*.[4] When we speak of Increasing Return in the present connection we are mostly not thinking of the concave portion of a curve

Mr. Flux, whose book has been cited above on behalf of the primary definition, seems to adopt the secondary one in his article on "Law" in Palgrave's *Dictionary*.

[1] In accordance with the Auspitz-and-Lieben construction, noticed in the ECONOMIC JOURNAL, Vol. XVII. p. 226.

[2] In the sense above explained. Good examples of (the materials for) such a curve are given by Cunynghame (*Geometrical Economics*); referred to by the present writer (ECONOMIC JOURNAL, Vol. XV. p. 67), and distinguished from a supply-curve, individual or other.

[3] *R* in Fig. 1 A, is meant to represent this point, corresponding to the *seventh* dose of labour (thirty-five days of labour), in accordance with the data of Table I. A.

In Fig. 1 B (the abscissa of any point on) *OS* may stand for the cost, the amount of outlay *x* multiplied by a constant; while the ordinate to the curve is the total yield (in money or other scale commensurate with the cost).

[4] For a more complete analysis the reader is referred to Prof. Pigou's stupendous article on "Producers' and Consumers' Surplus" in the ECONOMIC JOURNAL, Vol. XX. [Restated in *Wealth and Welfare*; with reference to which see II., 323, 433, and contexts.]

such as that in Fig. 1 A; but of something quite different, which might be illustrated as follows:—Let the curve in Fig. 1 A represent the cost-curve for an individual typical entrepreneur. With the increase of production in virtue of "external economies," the curve, or that tract of it with which we are concerned, may be lowered as a whole in such wise that each amount of product, x, corresponds to a lower cost. Similarly, Diminishing Return now has a signification other than the convexity of a curve such as that in Fig. 1 A.

It may be worth remarking that when we contemplate the working of a competitive regime as bearing on the interest of the community, from the point of view of the philosophic statesman, then we welcome the phenomenon of Increasing Return (or deprecate its contrary) as tending to (or from) some quantity which it is proposed to maximise.[1] But the criterion of such a maximum is analogous to our *primary* conception.

(*b*) When we leave the case of perfect competition, the sort of return which is diminishing in the primary but not the secondary sense—the (convex) tract of the curve in Fig. 1 A between the points P and Q—becomes more significant. Suppose that the *general* expenses of a Company, like that of the canals in France, were defrayed by the Government, then, even though the ruling price, determined, say, by competition, were inadequate to the total expenses, it might be the interest of the Company to produce an amount between (the amounts corresponding to) P and Q. Something similar occurs in the case of two competing railways obliged in the struggle for survival to leave out of[2] account past expenses of construction. As we continue to remove the conditions proper to the regime of competition, the importance of the point Q, at which Diminishing Return in the secondary sense sets in, becomes less conspicuous. Suppose that in the case put by Professor Carver[3] the farmer has a limited amount of capital-and-labour, say thirty-four days' labour, to apply to plots of land which he can have for nothing. The arrangement which he will find most profitable is to cultivate two such plots of land, applying to each seventeen days' labour; since thus on the assumptions of our Table II. he would produce (twice 317 =) 634 bushels; whereas, by applying the whole thirty-four days' labour

[1] J. S. Mill sometimes expressed himself as if the greatest *average* well-being was the *summum bonum*. But the better opinion, I think, is that of the philosophic Sidgwick that the end of political action is to maximise the *quantum* of happiness.

[2] The case of *duopoly*; below, p. 117.

[3] *Distribution of Wealth*, ch. ii.

to one plot, he would have produced less than 560 bushels (the produce of thirty-five days' labour according to Table I.).

In ordinary monopoly the outlay would not be limited thus absolutely, but by the necessity of limiting the production in order to keep up the price. The limit may be exhibited by one of Auspitz and Lieben's Constructions. In our Fig. 1 A let the curve represent cost to a monopolist of any amount z produced. And substitute (in imagination, not shown in the figure) for the straight line OS a curve passing through O concave to the abscissa; the ordinate representing the total value of the abscissa, z. Then the point of maximum profit to the monopolist may well prove to be a point in the tract between P and Q. Thus it by no means seems to be a universal truth that "with a given road-bed and with a given equipment in the way of depôts, offices, machine shops, etc., and with a given labour force, an increase in the rolling-stock will, between rather wide limits, enable the road to carry more freight and passengers; but this increase in its capacity will not be proportional to the increase in the rolling-stock." [1] If we represent the outlay on the "given road-bed" by OL in Fig. 1 A (not drawn to scale), and the outlay in rolling-stock by increments along OX above L, it is not certain that this outlay will be pushed up to a point corresponding to Q in the figure, as the above statement implies. If the demand for passenger-service is very inelastic, it might be the interest of the Railway to restrict the supply within such limits that the increase of carriage room would present Increasing Return (in the secondary sense contemplated). Nay, it is quite possible that Increasing Return in the primary sense may rule; the monopolist may arrest production at a point below even P in our figure,[2] a point beyond which he would lose by the falling-off of demand more than he would gain in cheapness of production.[3] This is possible but not probable. For there is a correlation—though not an identity—between the criteria of maximum

[1] Carver, *Distribution of Wealth*, ch. ii. p. 86. *Cf.* p. 88: "An increase of the rolling-stock would (except in very exceptional circumstances) increase, but not proportionately, the carrying capacity of the road."

[2] As pointed out by the present writer, ECONOMIC JOURNAL, Vol. XVII. p. 236.

[3] For instance, it is possible that railways in America are deterred from lowering passenger fares, not so much by the cost of increased accommodation as by the belief that the demand would not keep up (*cf.* Johnson and Huebner, *Railway Traffic and Rates*, Vol. II. p. 207). They may be wrong in that belief as Weyl and others urge (*cf.* Johnson, *American Railway Transportation*, p. 160); but it is possible that they may be right. But owing to the circumstance of Joint Cost (for freight and passenger) and Discrimination (between passenger fares) a clear-cut concrete example is not to be expected.

(1) for the net profit of the monopolist (affected by selling price), and (2) for the amount of product in kind (not so directly affected).[1] While the primary conception is thus less important in a regime of monopoly than in one of competition, the secondary conception is *much* less important. It is even fallacious. The suggestion which has been made by authors of note that the quantity which the monopolist seeks to maximise is the *average* return to his outlay—the rate, not the amount, of profit—is a misleading suggestion.

(c) The peculiarity of State Monopoly is that it seeks—to some extent at least[2]—to maximise not pecuniary profit in the ordinary sense, but the advantage of customers measured in money, the collective Consumers' Surplus. Now, the characteristic of such a maximum is a relation of the kind designated by the primary rather than the secondary definition.

Upon the whole, I think, there abide both the primary and the secondary definition; and the greater of these is the primary.

9. *Plural factors.*[3]—These comparisons may be transferred to the case of plural factors of production, *mutatis mutandis.*[4] The

[1] Let the net profit be $\zeta - x\pi$, where ζ is the money value of the product obtained by the application of the amount (in kind) x of the factor of production, π is the (supposed constant) price of the factor. Also $\zeta = zp$, where z is the amount of product in kind, and p is the price thereof (supposed liable to be varied by the monopolist):

$$\frac{d^2\zeta}{dx^2} = \frac{d^2z}{dx^2}p + 2\frac{dz}{dx}\frac{dp}{dx} + z\frac{d^2p}{dx^2}.$$

Accordingly, if $\frac{d^2z}{dx^2}$ is negative (Diminishing Returns in the primary sense ruling) *probably* $\frac{d^2\zeta}{dx^2}$ is negative. For the latter quantity is equal to the former ($\times\, p$) *plus* two quantities, one of which is known to be negative $\left(\frac{dz}{dx}\right.$ being positive, $\frac{dp}{dx}$ negative), while the sign of the other is quite unknown. The probability is of the kind which I have described as *a priori* in former numbers of the ECONOMIC JOURNAL (in particular, Vol. XX. p. 287–8). I should like to add to the instances there given Professor Pigou's (virtual) recognition of the principle when in his important communication to the Poor Law Commission (Appendix lxxx.) he argues that " unknown facts are as likely to conform as to conflict with known facts." [For further references see Index, s.v. *A priori Probabilities.*]

[2] In making this qualification I have in mind Dr. Marshall's doctrine of " compromise benefit."

[3] The plural factors are here considered as forming one product. The case of several products with several factors falls to be considered under the head of *Joint Production.*

[4] Among the changes required is the introduction of a new symbol, κ, to denote the sum of the money-values of the factors. By means of the production-function $z, = f(x, y, \ldots)$, where $x, y\ \ldots$ are amounts of the factors in hand, we may determine the values of $x, y\ \ldots$ which give the maximum value of Z, the net profit, $(= \zeta - \kappa)$ for any assigned value of κ; and thence obtain Z as a

required changes tend to augment the difference between the two definitions, to enhance the precedence of the primary. Beginning with two factors of production, let us measure their amounts in kind along two axes, OX and OY. And let a perpendicular to the plane of those axes (say the plane of the paper) at any point in the plane, designated by the co-ordinates x and y, represent by its height z the product in kind resulting from the employment of x and y in the best available ways. Then the terms Increasing and Diminishing Return are to be defined by the character of the surface which is traced out by z, when different values are assigned to x and y. According to the primary definition, returns are decreasing where the surface is thoroughly concave.[1] The secondary definition is something very different from this; more different than appeared while we were dealing with only one factor. Before, given a point x, we took a point below it, x_0, and compared the increment of produce due to the dose $(x_1 - x_0)$ with the increment due to the dose $x_2 - x_1$. Now, given a point (x_1, y_1), we have to take a point (x_0, y_0) (where one at least of the variables subscribed 0 is less than the corresponding variable subscribed 1), and to compare the increment of product due to the (compound) dose $(x_1 - x_0, y_1 - y_0)$ with the increment of product due to the dose $(x_2 - x_1, y_2 - y_1)$, where the points (x_0, y_0), (x_1, y_1), (x_2, y_2) are in a right line. For the special case in which the point (x_0, y_0) is at the origin $(x_0 = 0, y_0 = 0)$, the primary and secondary definition come to the same. But this is now a *very* special case. For there are any number of other lines, besides the one passing through the origin, which may be drawn (in the plane of x, y) through the point under consideration (x_1, y_1). It may well be that, by comparing increments corresponding to points on some line not passing through the origin, the surface may be shown to be *convex* in the neighbourhood of (x_1, y_1), though by the test of the " secondary " sort it appeared concave. Accordingly, I do not hold with the writers who attach a mighty importance to the question whether, if all the factors of production are increased in a certain proportion, say $\alpha : 1$ (where α is greater than 1), the product is, or is not, increased in that proportion. The matter has little to do with that character of the function z with which

function of κ as shown in a subsequent note. This function may be represented by the curve in Fig. 1 B (the abscissa now denoting the money-value κ). The reverse function which has κ as the dependent, Z as the independent variable—the cost-curve—may be represented by the curve in Fig. 1 A.

[1] See Index, s.v. *secondary*.

the entrepreneur is, and the economist should be, especially concerned, the fulfilment of the condition of a maximum.

Nor is the breach between the two definitions healed by taking for our lower point (x_0, y_0)—not $(0, 0)$, but—$(0, y_1)$ [or $(x_1, 0)$]; and observing whether (ax_1 with y_1) [or (x_1 with ay_1)] will produce more or less than a times what (x_1 with y_1) will produce.[1] According to both these subordinate varieties of the secondary species, as well as the preceding main one, it might appear that the surface at the point (x_1, y_1) was thoroughly concave; the sections of the surface formed by three vertical planes through the point (x_1, y_1) showing each a curve *concave* in that neighbourhood. And yet some other vertical section through the point might show a *convex* curve. Thus, if x_1 represent the number of cattle, y_1 the number of men attending to them, on a grass farm of given size, it is quite possible that each of the three variations (ax_1 with ay_1), (ax_1 with y_1), and (x_1 with ay_1), compared respectively with (x_1, y_1), might present diminishing returns in such wise that it would not pay the farmer to adopt any of these arrangements. And yet it might pay him to increase one of the factors a times, and the other β times, since the increment due to the change, compared, as it should be, with the cost incurred might well show an increasing return (in the primary and here essential sense). In the note will be found [2] an example in which, if a

[1] The phrase what ax with y will produce (or what ax with ay will produce), borrowed from Professor Carver (*loc. cit.*), is used as the equivalent of $f(ax, y)$ (or $f(ax, ay)$).

[2] What concerns the entrepreneur is the sign of $\frac{d^2\zeta}{d\kappa^2}$, where ζ as before is the gross yield in money and κ is the cost in money (say $x\pi_1 + y\pi_2$, where π_1, π_2 are the—supposed constant—prices of the factors). This comes to the same as $\frac{d^2z}{d\kappa^2}$, where z as before is the product in kind and $\zeta = zp$, if p may be treated as a constant.

For example, suppose that initially the amounts of the two factors are $x = 2$, $y = 1$. And let the law of production, in the neighbourhood of these values, be such that the gross profit ζ $(= zp) = 9x - 5y - 3x^2 + 4xy - y^3$. Which is also $= z$, the product in kind, if p, the price of the product, is unity. Also let the outlay on the factors, $\kappa = x\pi_1 + y\pi_2 = x + y$; the price of each factor being unity.

Initially—when $x = 2$, $y = 1$—the product (and gross profits) are 8. Now compare this with the product of ax with ay, where a is $\frac{3}{2}$. That product proves to be 8·25, *less* than $\frac{3}{2} \times 15$. A like result follows for values of a less than $\frac{3}{2}$. Likewise if the product of ax with y, or of x with ay is compared with the product of x with y, diminishing returns in the proposed sense are shown. And yet the returns are increasing in a sense with which the entrepreneur is principally concerned. For putting Z for the net profits $(= \zeta - \kappa)$, find the locus at which Z is a maximum for any assigned value of κ. Considering a line $x + y = \kappa$ in the neighbourhood of the initial point (at which $x = 2$, $y = 1$, $\kappa = 3$), find the value of x and y which maximise Z, that is, $8x - 6y - 3x^2 + 4xy - y^2$

is $\frac{5}{4}$, β is 2, Increasing Return is shown; though if $\alpha = \beta$, or if either of them is 1 (unity), the case seems to be one of Diminishing Return.

A more practical instance could no doubt be attained from the business of railways, or Trusts; assuming (as may often, I think, be legitimately assumed) that the directorate is not deterred from offering additional services to the public by fear of demand falling off.

The divergence which has been indicated between the primary definition of the terms in question and that which is suggested by the semi-mathematical treatment of the subject becomes aggravated when, instead of dual, we have plural factors.

The property of plural factors which has been pointed out, that in starting from any point (system of factors) there is a choice of directions, is connected with the property that in moving from any initial point to the position of maximum, there is a choice of paths.* By the purely mathematical economist the free path would be conceived as movement in that direction by which the greatest increment of profit is continually obtained, the *line of preference* (perpendicular to the line of indifference).[1] But it is not possible, I think, to say *a priori* which of various types best represents the working of the managerial mind.

10. *Relative discontinuity.*—The analysis of different paths, different sequences of steps by which a business may be extended, brings into view the circumstance that one factor of production

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Substituting $\kappa - x$ for y, we find that the resulting expression in x becomes a maximum when $x = \frac{3}{8}\kappa + \frac{7}{8}$; corresponding to $y = \frac{5}{8}\kappa - \frac{7}{8}$. Substituting these values of x and y in the expression for ζ we obtain for the required locus ζ an expression representing a curve convex to the abscissa, κ (like the curve in Fig. 1 B up to the point P); corresponding to Increasing Return certainly in the primary, and quite possibly also in the secondary sense.

These results are independent of the assumption, which has been made for convenience, that the prices of the product and of the factors of production are each unity. The results depend upon properties of the production-function which do not vary with the price: in particular the *saddle-shaped* character of that surface in the neighbourhood with which we are concerned. The secondary test is so deceptive because there is not fulfilled the condition proper to the primary test, the condition of a maximum:

$$\left(\frac{d^2f}{dx^2}\right)\left(\frac{d^2f}{dy^2}\right) > \left(\frac{d^2f}{dx\,dy}\right)^2.$$

The only assumption made is the constancy of the prices of the factors while the amount of production is varied. If that assumption is not admissible, the significance of the primary definition, the insignificance of the definition in terms of proportionate factors are still demonstrable.

* There are here omitted some passages employed in the original to illustrate mathematically the different courses by which an entrepreneur may vary plural factors of production so as to attain the most advantageous combination.

[1] See *Mathematical Psychics*, p. 22.

often varies discontinuously as compared with another. Two factors do not always move continuously like the minute hand and the hour hand of an ordinary clock. The movement is rather like that of a clock invented by the ingenious R. L. Edgeworth, in which one wheel moves with every swing of the pendulum, while another connected with the (hour) hand of the clock moves *per saltum*, the hand taking a jump every 7½ minutes. Or we may liken the discontinuous variation to a flight of very steep stairs; like the steps in the Great Pyramid, each some four feet high.

Take as an example of ledges surmounted by large steps or jumps, plots of land not divisible below a certain minimum, and for the more finely graduated steps, days' labour applied to a plot of land, as in the example above quoted from Professor Carver.[1] Starting from the zero of outlay, we find (i.) for the first plot the cost in money of producing a certain number of bushels z to be of the form l, the fixed rent of the plot, $+ \mu$, an outlay varying with z. If, as in the case supposed—the case typical of the industries here contemplated—the land without labour produces nothing, the cost-curve must start as in Fig. 1 A at a point, L, on the axis of OX, above the origin. It follows that initially Increasing Return in the primary sense must rule; whether or not the curve traced by x — and ξ, the money value of x — is concave.

Suppose, now (ii.), that labour and capital has been laid out on the first plot up to the point of maximum profit R; and that a second plot of land is then taken on. *Ceteris paribus*, and in particular the price of the product being supposed constant, it may be shown that the outlay of " days' labour " on the second plot will present Increasing Return in the secondary sense. And so on for additional plots. It is more to the purpose, I think, to observe that the outlay on labour and tools for the last plot taken on will present Increasing Return in that primary sense in which the first of the two compared doses is reckoned from the beginning of the outlay on the last plot. This follows by a repetition of the reasoning applied to the first plot.

The case considered in the two preceding paragraphs is very important, so important as often to obtain the title of Increasing Return *par excellence*.[2] It plays a great part in the theory of Railway Rates. It is the *rationale* of the often noticed circumstance that an increase in the gross receipts is apt to be accom-

[1] Above, p. 63.
[2] *E.g.* Hadley, *Economics*, p. 154, *note*.

panied with a more than proportionate increase in net receipts. To establish this and other important conclusions, it would be often, I think, unnecessary to take into consideration, as just now, a *second* jump, or large dose. For instance, when Professor Ripley employs the principle to account for the prosperity of the American railways consequent on the increase of business,[1] he may be understood, I think, as regarding the outlay on the construction and maintenance of the railways as corresponding to our l [in the simpler case above labelled (i.)], and the operating expenses as corresponding to our μ.

As costs corresponding to our μ *initially*, after the *per saltum* variation of general expenses corresponding to our l, afford increasing returns (in a certain sense), so *ultimately*, if μ is continually increased while l remains constant, diminishing returns must set in. This corresponds to the fact made familiar by Dr. Marshall that when an industry is called for a sudden increase of output, the short-period supply curve is apt to be inclined positively.

Having now defined the law of increasing (or diminishing) return considered as a term or conception,[2] let us go on to consider the propositions into which that term enters.

11. *The laws as propositions.*—One class of propositions connects the terms defined with peculiarities in the incidence of taxation. But these are not the propositions usually understood by the laws now under consideration. Rather the attribute that is connected with the quantitative relation above defined is the cause of that relation. Thus, in the statement above[3] quoted from Dr. Marshall, the character of Diminishing Return is connected with conditions of agriculture. It is objected by some that the causes are too diversified to allow us to speak of a law of return. Let us consider this objection with respect to each of the laws separately.

12. *Law of diminishing return.*—The peculiar significance of this law in agriculture has impressed a high authority, Mr. Bullock,[4] so much that he proposes to restrict the law to "the productivity of labour on a definite tract of land." Of course, now

[1] Report of the United States Industrial Commission, Vol. XIX. *Transportation*, pp. 277, 286–7. Similar expressions as to the nature of Increasing Return might be quoted from other railway experts.

[2] Some authoritative uses of the terms, presenting points not very directly related to railway economics, will be examined in an Appendix (below, p. 95).

[3] Above, p. 62.

[4] *Quarterly Journal of Economics*, Vol. XVI. (1902), p. 484, and context.

that the law is considered as implying a cause, it is quite legitimate to give a narrow interpretation to the always somewhat arbitrarily limited word " cause." Thus, the laws of the tides according to Sir G. Darwin, who recognises that " a true tide can only be adequately defined by reference to the causes which produce it," [1] presumably relate only to the periodical oscillations of the sea caused by the moon and the sun, but not to those caused by periodic winds, variation of atmospheric pressure, etc. However, it is " practically convenient " to speak of those changes as " meteorological tides." [2] A like use of qualifying adjectives—recommended by Dr. Marshall as suitable to economics—might remove Mr. Bullock's scruples. That they are not obstinate scruples I infer from the author's use of the terms in a work subsequent to that above quoted.[3]

The diversity of cause may appear greater in the case of Diminishing than in that of Increasing Return, if with Professor Seligman we include the Law of Diminishing Utility under that of Diminishing Returns. And certainly it is not easy to keep the two laws quite separate; especially if the former includes increasing disutility, affecting the cost of labour.[4]

On the other hand, there is in one respect a greater unity in the action of Diminishing Return—that it always rules, *provided that we take sufficiently large doses*. In the nature of things the function representing net advantage cannot increase indefinitely as the factors of production are varied; its value must ultimately pass through a point of maximum—a *Wendepunkt*. This circumstance, it should seem, has so impressed one whose impressions deserve attention, that he regards the law of diminishing return as " no more than an axiomatic statement of a universal principle " . . . " an axiomatic and sterile proposition." [5] My criticism of

[1] *The Tides and Kindred Phenomena*, p. 2. [2] *Loc. cit.*

[3] *Elements of Economics* (1905), ch. v. § 44. " What is true of land is true also of labour and capital."

" Beyond this point [" twenty acres which will perhaps yield the largest return obtainable from the labour of one man "] it will not pay to invest land and capital if the services of only a single worker are available; so that we find here diminishing returns to investments of land and capital with a given supply of labour."

[4] I accept Dr. Marshall's distinction : " The tendencies of diminishing utility and diminishing return have their roots, the one in qualities of human nature, the other in the technical conditions of industry " (*Principles*, ed. vi. p. 170, note); and I evade the difficulty that the price of labour (which enters into Net Returns) has roots in the qualities of human nature, by treating the price of labour along with that of other factors of production as a constant—as I think it is allowable when treating of rates and fares, but not wages and salaries.

[5] Wicksteed, *Commonsense of Political Economy*, pp. 529, 530.

this pronouncement may be expressed in terms of a metaphor which I have already employed, the representation of net profits by the height of a mountain-shaped surface above the plain. If an Alpinist, with a view to climbing up to the summit, seeks to ascertain the configuration of the surface of his immediate neighbourhood; what are we to think of a guide who protests that there is no need for anxious inquiry whether the surface is concave or convex, for (as no mountain rises to heaven) a sufficiently long step must always lead downwards; that, therefore, it is not only true but a truism that the surface with which the Alpinist is concerned is *concave* (viewed from below).

With respect to Diminishing Return in the sense which is of practical interest in industry generally, I think we may say that the phenomenon has all manner of causes [1] except or besides those botanical ones which are characteristic of the law in its first and still most important form relating to agriculture.

13. *Law of Increasing Return.*—I shall now attempt to summarise the various conditions which are attended with the attribute Increasing Return in one of the senses above distinguished.

(1) First may be placed the circumstance which Mill places first,[2] that some things in order to be produced at all must be produced on a large scale—a railway, for instance. Here the outlay up to the large minimum requisite to produce any return at all may be considered as producing no return; and accordingly the cost-curve corresponding to *OLPQR* in our Fig. 1 A,[3] starts from the origin as a vertical line, which we have seen involves the character of Increasing Return (at least, in the secondary sense).

(2) Next I would place the general principle that size is favourable to multiplication of parts, and so to "co-operation" in the sense in which the term is employed by J. S. Mill after Wakefield, "organisation," as it is now usual to say, "differentiation-and-integration" in the technical phraseology of Herbert Spencer,[4] "system" in the language of good old Bishop Butler.[5]

[1] Professor Seligman has well illustrated the variety of causes leading to a similar result in different departments of production :—

"If we crowd more people into the same omnibus, or run more trams over the same track, or make the labourer tend more looms, or put more manure into the same field, we have a more intensive utilisation, until finally the intensive margin is reached where the additional returns will not compensate the additional effort or outlay."—(*Principles of Economics*, § 88 and context.)

[2] First among the advantages of the Joint-stock principle, *Political Economy*, Book I. ch. ix. § 2.

[3] Above, p. 64.

[4] Employed by Dr. Marshall.—*Principles*, Book IV. ch. viii. § 1.

[5] Preface to *Sermons;* and *cf.* note to Sermon on the Ignorance of Man.

Mangoldt [1] and F. B. Hermann [2] may be referred to as putting the matter particularly well.

(3) Where there is a co-ordination of several parts or factors, it often occurs that one varies discontinuously as compared with another, in the manner above illustrated.[3] Foremen and the workpeople whom they supervise may be instanced. When of r foremen each has the full complement of workpeople to which he can attend with advantage, if an $(r + 1)$th foreman is taken on, Increasing Return acts before the foreman last taken on has his full complement of men.[4] There occurs the gain described by Babbage and J. S. Mill as employing the workpeople, and likewise the machinery, up to their full capacity. This advantage is what Jevons [5] designated "Multiplication of Services," attributing its first enunciation to Archbishop Whateley. As we have seen, this principle is a main cause of increasing returns in the railway industry.

(4) Next I should place the classical trio of advantages attributed by Adam Smith to Division of Labour. Their importance is diminished indeed, but not abolished by modern conditions. Practice makes engine-drivers, as well as pin-makers, perfect. There is, I suppose, less "sauntering" on the part of porters at a large than at a small station. The invention of distant signalling by a points-man [6] who sought to spare himself trouble may match Adam Smith's example of his third advantage.

[1] *Grundriss*, § 29.

[2] *Staatswirtschaftliche Untersuchungen*, p. 217, ed. 1890. One of Hermann's examples, the extinction of a fire, illustrates training ("eine eigens geübte Mannschaft") as well as system. The latter advantage in its purity may be illustrated by a primitive version of the example. Once upon a time—soon after the invention of fire, perhaps—there was a conflagration to extinguish which a number of men carried buckets full of water from a neighbouring stream to the scene, of the fire. Then supervened an organising intelligence directing the men to stand in a row at a distance of a couple of yards or so from each other and to pass the buckets without moving from their respective places. Thus the labour of a number of men running to and fro (or at least the excess of the original leg-work over the substituted arm-work) was saved by mere organisation: nothing but an idea (and the numbers requisite for its realisation). This kind of Increasing Return is too much ignored by writers on Railway Economics who dwell exclusively on our *third* type (*cf.* ECONOMIC JOURNAL, Vol. XXI. p. 370).

[3] *Loc. cit.*, p. 78 *et seq.*

[4] For a more exact statement see below, subsection 20, on prime cost; where in general for πx should be substituted μ, a function of x (as in the parallel passage, *loc. cit.*, p. 369); a function which is zero when x, the amount of product due to an additional dose of the discontinuous variable (*e.g.* an additional foreman) is zero, and small when x is small, and so secures the fulfilment of Increasing Return (in a certain sense), initially at least.

[5] *Economic Primer*, p. 35.

[6] Findlay, *Working and Management of an English Railway.*

(5) Follows a large miscellaneous class of advantages more or less cognate to the above, enunciated by a variety of eminent writers, from Plato to Marshall.

To head (1) we may refer an advantage such as that which is enjoyed by large ships in respect of resistance to the water.[1]

May we not refer to the same head the stimulus which the presence of numerous fellow-workers imparts to the performance even of unco-ordinated operations ? [2]

Should we connect with head (2) [and head (3)], the Platonic *καιρός*, the power of utilising opportunity attributed to Division of Labour ? This advantage may be illustrated by the efficiency of a Fruit-Car Line which lends its services to different railways in districts so diverse that the respective fruit-crops ripen at quite different periods—peaches in Georgia after the middle of June, citrous fruit in California between December and May. The large private line supplies refrigerator cars in sufficient number to transport the whole of each crop in due season. Whereas, if each railway were itself to collect the fruit in its own district, it would not have cars enough to transport the fruit at the seasonable moment (or would have at other times to keep them idle).[3]

May we refer to some of the other heads an advantage which some may think ought to constitute a separate head, and, indeed, the first head ? It is, indeed, first historically, as it comes first in Plato's enumeration; and it is not last in importance. This is the giving to each the task for which he is best fitted; classifying the work-people according to their capacity.[4] But may not this advantage be subordinated to our head (2), together with head (3), the advantage of employing differentiated organs to their full capacity ?

Shall we refer to the same heads the advantages procured through "integration," when all the processes of production, from the raw material to the finished product, are performed by the same firm—especially when there is advantage not only to the large firm, but to society, by dispensing with "the need of maintaining too many selling agencies" ? [5]

With head (3) may we connect, as Jevons does, what he calls the "Multiplication of Copies" ? [6] And may we not refer to the

[1] *Principles of Economics*, ed. vi. p. 290.

[2] *Cf.* E. Wakefield (*Ireland*), 1812).—"Irish labourers never work singly . . . the people there have a sympathy of feeling which makes company necessary for those at work."

[3] Johnson and Huebner, *Railway Traffic and Rates*, p. 234.

[4] *Cf.* Mill (*loc. cit.*) quoting Babbage.

[5] Hadley, *Economics*, p. 154.

[6] *Economic Primer*, p. 36.

same head the advantage of Interchangeable Parts, on which Dr. Marshall has particularly dwelt?

I do not attempt a full enumeration, nor do I insist on the logical affiliation which I suggest. I am only concerned to point out that there is a certain thread of connection holding together the majority at least of the advantages which have been enumerated. There is only one cause, I think, which lies apart from the others; and it is one, I think, usually, perhaps properly,[1] not included among the causes of Increasing Return. That is the greater stability of a large business, the connection between magnitude and the principle of insurance.[2] It was *a priori* improbable, and ultimately proved false, that all Antonio's multifarious ventures should have failed :—

> "From Tripoli, from Mexico and England,
> From Lisbon, Barbary and India."

A large railway, serving varieties of industry and pleasure-seeking, is less likely to lose its custom than a small line which depends on one class of custom.

In the railway industry the practical importance of the cause above placed third no doubt deserves the pre-eminence assigned to it by experts; but in a philosophical view of the subject it should be recognised that there are many other causes about as operative in the railway industry as in other departments of production.

I go on to the cognate conception of Joint Cost.

14. *Joint Cost.*—This conception is not only cognate, but even coincident, with that of Increasing Return, according to one of the parties in a battle of giants which has been fought in America over the matter.[3] I agree with Professor Taussig that there are two distinct conceptions; but I concede to Professor Seligman that they have a certain attribute in common, and that the cases which they denote are frequently coincident.

(1) The two terms, as I understand, correspond to two distinct

[1] For the increasing return attends the large scale only provided that the large scale is attended with a plurality of causes that are *independent* in the sense appropriate to the theory of Probabilities, and the proviso sometimes fails (as probably in the case of some Trusts).

A like objection might, however, be made to the attribution of Increasing Return under other heads. For instance, increase of size is not *necessarily* attended with increase of "differentiation" (above, subsection 2).

[2] J. B. Clark was, I believe, among the first to point out the advantage of large concerns in this respect (*Quarterly Journal of Economics*, 1892).

[3] See *Quarterly Journal of Economics*, Vol. XX. p. 631, and Vol. XXI. p. 156; referred to by Mr. Maurice Clark in his *Local Freight Discrimination*, pp. 27, 28.

mathematical conceptions, which may be simply presented as follows :—Let x and y be the respective amount of two joint products [1]; and let z be the cost of production, upon the usual supposition that the factors of production are employed according to the best available methods.[2] Then if P is a point in a plane, say the plane of the paper, of which the co-ordinates are x and y, z may be represented by the height of a surface above the point P. For example, let the surface be represented by that of a half-orange placed on the plane of x, y so that the base of the hemisphere is coincident with the circle $XOYo$ in our Fig. 2. Now through any point P (within the square $BCAO$) hold a knife parallel to the line OY and perpendicular to the plane

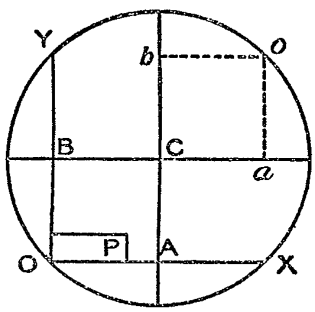

Fig. 2.

of the paper, and cut or suppose cut, a thin slice or section of the orange. The curve which bounds that section will represent the (total) cost of producing *any* value of y together with the given value of x, the abscissa of P. As the value of y increases (the value of x remaining constant), the corresponding value of z increases,

[1] The reader will observe that these symbols are not used here in the same sense as in former paragraphs (above, p. 65 *et seq.*). The familiar x and y being now reserved for the quantities on which it is now desired to fix attention, the joint products, let us put u and v for the factors of production, at first supposed two in number. Let $x = \phi(u,v)$, $y = \psi(u,v)$; and accordingly $u = f_1(x,y)$, $v = f_2(x,y)$. If, as before, the respective prices of the factors are π_1, π_2, we have $\pi_1 f_1(x,y) + \pi_2 f_2(x,y) = z$ (corresponding to κ in former paragraphs). If p_1, p_2 are the respective prices of the products, we have $p_1 x + p_2 y - z$ for the net product which is to be maximised. If p_1 and p_2 are treated as constant, z must fulfil the conditions of a *minimum*.

If there are more than two factors, the case is to be treated on the analogy of a single product with plural factors of production (above, p. 74 *et seq*). If there is only one factor, the case degrades to the rudimentary type defined by Mill, as mentioned in our text (below, p. 88).

[2] Above, p. 69, note 4.

but at a diminishing rate. For the *slope* of the curve[1] continually decreases as y is increased, it is less for the point (of which the co-ordinates are) x, $y + \Delta y$ (where Δy is a small increment or "dose" of Y) than what it is for the point x, y. This relation between contiguous points we know to be a condition of Diminishing Cost or Increasing Return. The relation is quite distinct from the relation between the said slope at the point x, y, and the similarly defined slope at the point $x + \Delta x$, y. This latter relation is the criterion of Joint Cost or Joint Production.[2] If, as we change from x to $x + \Delta x$, *ceteris paribus*, the increment of z due to an increment of y becomes smaller, this means that an increase in the production of the commodity represented by the abscissa (x) makes it less costly to increase the production of the commodity represented by the ordinate (y).[3] In the example given—a hemispherical surface—Increasing Return and Joint Cost go together. But it is easy to imagine a surface—that of a melon, for instance[4] —for which the contrary is true. Even the simple example which we have given suffices to show that the two characteristics are not identical. If P be taken very near CA while far from CB, it may be shown,[5] and is perhaps self-evident, that the slope in question will decrease very slightly in consequence of an increase

[1] The "slope" of a curve at any point therefrom is here used to denote the tangent of the angle made with the abscissa by a tangent to the curve at that point. For the curve under consideration (in a plane perpendicular to the plane of the paper) the line through P perpendicular to OX (in the plane of the paper) is to be taken as the abscissa.

[2] I use the two terms to denote (different aspects of) the same phenomenon.

[3] In symbols, $\frac{d^2z}{dx\,dy} < 0$; where differentials do duty for finite differences, increments which may have different magnitudes according to the context and purpose. More exactly the characteristic (of Joint Production) may be written :—$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y) < 0$; where $f(x,y)$ denotes the cost of producing the joint products x and y.

[4] See below, p. 89, note 1.

[5] The equation to the surface may be written

$$z = \sqrt{2c^2 - (c - x)^2 - (c - y)^2};$$

where c is put for the radius of the circle divided by $\sqrt{2}$. We have then for $\left(\frac{dz}{dy}\right)$, the differential coefficient of z with respect to y on the supposition that x is treated as a constant, $+ (c - y)/z$, which is positive, provided that y is less than c, as postulated in the text. The differential coefficient of this expression with respect to y while x remains constant, say $\left(\frac{d}{dy}\right)\left(\frac{dz}{dy}\right)$, is $\frac{-z^2 - (c - y)^2}{z^3}$; which is always negative and generally considerable. But $\left(\frac{d}{dx}\right)\left(\frac{dz}{dy}\right) = \frac{-(c - x)(c - y)}{z^3}$; which also is always negative, but becomes evanescent as x approaches c.

of the abscissa (alone); while as before it decreases sensibly in consequence of an increase of the ordinate (alone).

The converse relations of Diminishing Return and the usually unnamed opposite of Joint Production which I have proposed to call Rival Production [1] may be illustrated by the half-orange, if we reverse its position so that what was before its highest point is now its lowest, the point C at which the surface now touches the plane of the paper. If C is now taken as the origin it will appear that the slope with which we are concerned will *increase* in consequence of an increase of either the abscissa or the ordinate (for any point p within the area $Caob$). Diminishing Return and the opposite of Joint Production go together. Such consilience is quite common. But it is by no means universal. For example, honey and certain fruits are, I believe, joint products; the flowers which produce the fruit being fertilised by the bees which produce the honey. But though Joint Cost thus operates, it is quite possible that an increase of fruit trees *ceteris paribus* would be attended with Diminishing Return. And the increase of hives *ceteris paribus* may have a like result.

These propositions remain true when we remove the clause "*ceteris paribus*," and consider Diminishing Return in its most general and genuine signification as equivalent to the condition or criterion of a *maximum*.[2] The circumstances may be such that in whatever proportions we increase the factors of production, bees and fruit trees, each successive increment of cost will be attended with a less than proportionate increment of produce.

(2) The two conceptions are clearly distinct. But though not coincident they are cognate. There is a certain general resemblance between Increasing Return and Joint Production in so far as both seem to fulfil the dictum: "Unto him that hath shall be added." More exactly, the resemblance may be traced with respect to one or more of three distinct features.

First (*a*), there is a certain correlation between the character of Increasing Return in the proper sense of the term and that of Joint Productivity in the sense above explained. The greater the Productivity the more probable it becomes, other things remaining the same, that the case will be one of Increasing Return. If in the example just now adduced we suppose the stimulus to the creation of honey given by the increase of fruit trees to become indefinitely greater while other features of the case remain

[1] ECONOMIC JOURNAL, Vol. VII. p. 54, referring to *Giornale degli Economisti*, 1897. "Disjunctive" might be suggested as the antithesis to "Joint."

[2] See ECONOMIC JOURNAL, Vol. XXI. p. 357, p. 364 and context

the same, the case will become ultimately one of Increasing Return.[1]

This is, I think, the most general view of the correlation between the two conceptions. But there are other kinds of consilience which depend upon some particularity in the function which expresses the relation between cost and products.

(*b*) *Pro forma* I begin with the simplest case which Mill begins with: "when the same outlay would have to be incurred for either of the two [products] if the other were not wanted or used at all."[2] Supposing that increase of one commodity x is always attended with the increase of the other in some definite relation,[3] the two characteristics will evidently concur.

A more important case arises when the Joint Cost depends upon a quantity such as total weight or volume which is the sum of two or more items each pertaining to one of the Joint Products.* The cost of carrying gold and silver, for example, might depend only on the avoirdupois weight, in a primitive regime, making abstraction of "general" expenses. There might be a co-operation[4] between native bearers such that an increase of the burdens would not require a proportional increase of men. Increasing Return would then be realised, together with Joint Production. Nor is it necessary to suppose that the contributions of the two articles to their Joint Cost are simply proportional to the respective weights. Differences of specific gravity (affecting the relation of volume to weight) or of value (affecting the amount of insurance) might be relevant. And yet upon probable assumptions, which it

[1] Let x denote the amount of fruit produced, y the amount of honey; and let the total cost of production be $f(x, y)$. In order that Joint Production, in the sense in which the term is here taken, may obtain, $\frac{d^2f}{dx\,dy}$ must be *negative*. In order that Diminishing Return, in the proper "primary" sense of the term, may obtain, we must have not only as indicated in the preceding paragraph of the text

$$(1)\ \frac{d^2f}{dx^2}\ \textit{positive, and}\ (2)\ \frac{d^2f}{dy^2}\ \textit{positive};$$

but also (3)

$$\frac{d^2f}{dx^2}\frac{d^2f}{dy^2} - \left(\frac{d^2f}{dx\,dy}\right)^2\ \textit{positive.}$$

The last condition will be violated if *ceteris paribus* the value of $\frac{d^2f}{dx\,dy}$ is supposed to increase indefinitely (in absolute magnitude).

[2] *Political Economy*, Book III. chap. xvi. § 4.—Mill begins with this case but he does not end with it. He continues: "In a more partial sense, mutton and wool are an example, beef, hides, and tallow. . . ."

[3] $y = \phi(x)$, while $\frac{d\phi}{dx}$ is continually $+$.

* See Index, s.v. *Joint Production*, for further reference to this case.

[4] Increasing return of our species 2, above, p. 81, unmixed with species 3.

suffices to indicate in a mathematical note,[1] Increasing Return and Joint Production would be consilient.

But this is not the leading case of correlation between the two compared conceptions. That is to be found rather (c) in the circumstance that frequently joint products depend upon one and the same factor of the kind above described as varying *discontinuously*, like the amount of land (relatively to the labour employed thereon) in the illustration given. Thus the carriage of a passenger at a time when there is no public service presupposes the running of a special train. Accordingly, the transportation of the passenger and that of luggage (at the specified time) are joint products. For—starting from zero—it is impossible to increase the one kind of transportation without rendering the other less costly.[2] In the same circumstances the transportation of a *second* passenger is attended with decreasing cost. So, considering the general expenses of permanent way and staff, we shall often find that passenger traffic and goods traffic are joint products. The arrangements necessary for a certain [3] increase of one facilitate the increase of the other. The case then is one of Joint Production. But it is also apt to be a case of Increasing Return. For, as shown above,[4] in these circumstances Increasing Return, at least of the *secondary* kind, is apt to be realised.

Conditions of this kind I think are usually presupposed by the able writers who identify Increasing Return and Joint Production. Thus Mr. Bickerdike in the following passage employs the term " scale of production " in a sense which seems to imply the sort of discontinuity which I have all along in view :—

[1] Let the Joint Cost, $z, = F(\alpha + \beta)$, where $\alpha = \phi(x)$ and $\beta = \psi(y)$. Let F' be continually +, and F'' — ; and let ϕ and ψ have the same properties. Then it may be shown that $\frac{d^2F}{dx\,dy}$ is negative, as well as $\frac{d^2F}{dx^2}$ and $\frac{d^2F}{dy^2}$. If F'',ϕ'',ψ'', are each > 0, *ceteris paribus*, then the opposite of Joint Cost, Disjunctive Production, co-exists with Diminishing Returns in that proper sense which is defined by the *three* conditions (characterising the second term of variation) of a maximum (or minimum). The theorem admits of generalisation in several directions. Thus z may be $= z_1 + z_2 + \ldots$, where each of the subscribed " z's " have the properties before attributed to z. Also here, as throughout, what is predicated of two variables is capable of extension to the case of several variables.

This principle accounts for the consilience between Increasing Return and Joint Cost which we observed in the case of our spherical orange. If we had taken a melon, attributing to it the shape of an ellipsoid of revolution, the consilience would no longer hold good; z now involving not only a *sum* of functions, such as $ax^2 + by^2$, but also a *product*, such as $2hxy$.

[2] ECONOMIC JOURNAL, p. 369; and *cf.* p. 368, and p. 367, par. 1.

[3] *Cf.* below, p. 94.

[4] *Loc. cit.*, p. 88.

"I have spoken of the law of increasing returns and of joint costs as the bases of justification for differential prices,[1] but it would be more correct to say a condition of production such that an increased supply of certain articles or services would make easier an increase of supply of other articles or services, of a further increase of the supply of the same articles or services. That is to say, if cost of production of x_1, x_2, etc., units, either of different commodities or even of the same commodity supplied to different customers, depends not only on x_1, x_2, etc., but also on the total scale of production Z $(= x_1 + x_2 + \ldots)$, *i.e.*, on $F_1(x_1,Z)$, . . ."

The symbolism proposed by Mr. Bickerdike appears to imply not only the discontinuity which we are now considering, but also the simplicity which was considered in a preceding paragraph. The *addition* of the joint-products, x_1, x_2, suggest their having in common a measurable attribute, such as weight or volume. The symbols must, however, be interpreted in a somewhat forced sense,[2] so as to apply to joint products like goods and passengers, which cannot well be added together. But the use of a simple symbolism in a loose sense is the more defensible because, however complicated, mathematical expression can hardly cope with the difficulties caused by the element of Time in economics.[3] That is the justification of our attempt to eke out the deficiencies of formal exposition by means of homely metaphor.[4] Even this resource fails us now; as we should require a *fourth* dimension to represent two factors of production and two products.

[1] The circumstances that both Joint Cost and Increasing Return are favourable to Discrimination is regarded by Mr. Maurice Clark (*Local Freight Discrimination*, p 27) as a reason for their identification. It is not a decisive reason in the view of one who does not regard Joint Cost as the fundamental cause of Discrimination (see ECONOMIC JOURNAL, Vol. XX. p. 460; Vol. XXI. p. 148, and sect. ii. [in the sequel] of the present paper).

[2] I have taken the liberty of placing a comma after x. I might suggest using a semicolon and writing $F(x_1;\ x_1, x_2 \ldots)$ where the symbols on the right of the semicolon are to be understood as varying discontinuously; *e.g.* by degrees corresponding each to a train-load or other relatively large unit. Thus if x_1 denote (the number of) third-class passengers, x_2 that of first-class passengers, the x_1 on the right together with the x_2 may determine the number of (daily) trains on a given railway; while the x_1 on the left denotes the number of third-class passengers on a particular train. The function F assigns the law of cost for the "short period" during which it is proper to treat the symbols on the right as constant, while the symbol (or it might be symbols) on the left of the semicolon are varied. [Compare the symbolism proposed, 1908.]

[3] *Cf.* Preface to *Principles of Economics*, "the element of Time which is the centre of the chief difficulty of almost every economic problem."

[4] *Loc. cit.*, p. 78.

It may be worth observing that the factor here described as varying discontinuously is not necessarily, though it is frequently, prior in time to the continuous factors. Suppose the factor to be the irrigation of crops, as practised by the Virgilian husbandman, who some time after sowing the seed admits the fertilising flood (" semine facto " . . . " *Deinde* satis fluvium inducit ").[1] If the operation of " enticing " the river from its channel could be performed only once in a year or other period during which it was open to the husbandman to plant and dig to any extent, then Increasing Return—of the secondary species, up to a point—would be realised. If there are two plots of ground suited to different crops, and the opening of the sluices to irrigate one plot involves the irrigation of the other, then the crops will be joint products.

In most of the examples which have been given it will be apparent that Joint Production (and its contrary) resembles Increasing (and Diminishing) Return in this respect: that as each is characterised by an increment relative to a dose,[2] so the character may vary with the magnitude of the doses contemplated. To predicate Joint Production without this datum would often be unmeaning. Thus passenger traffic and goods traffic may be considered as joint products with respect to variations on a large scale involving the construction of a new track; both kinds of traffic being thereby facilitated. But for a given track, which is already crowded, an increase of one kind of traffic may well render the other kind of traffic more costly.[3] The case may be one of *rival* production. The complexity of the facts with which the railway manager has to deal transcends the nicety of the nomenclature invented by the economist.

15. *Prime Cost.*—One more cognate term, one more class intersecting those which have been defined, remains to be considered. The affinity or partial coincidence of Prime Cost with the preceding categories is, indeed, not very evident if we identify the term with what is sometimes called the " special cost " of a product, meaning that cost[4] which would have been saved if

[1] Georgic I. 106. For a more modern instance, see below, p. 94.

[2] More exactly the increment of a differential coefficient. See as to the significance of size, subsection 3, *loc. cit.*

[3] Mr. Acworth describes instructively the impediment to through traffic on a railway which might result from the stimulation of short-distance coal traffic. *Railways and Traders*, pp. 124, 126.

[4] I might suggest assigning this signification to the term " special " cost, and to " prime " cost the somewhat different signification proposed in the text. No doubt the line of distinction is very fine. For the special cost of the local freights instanced in the text would be a kind of " prime " cost in so far as those

the product had not been produced. For example, if there were proposed a revision of railway rates for local traffic (extending, it may be, to a great many localities—a large part of the system), "in figuring whether the new rates would be good financial policy, the road must charge against the traffic as its 'special cost' every expense that can in any way be causally traced to the local freight traffic. This means that large items of maintenance, interest on cost of rolling-stock, and structures, etc., etc., must be included." [1]

Such computations must certainly be made by the entrepreneur varying the factors-of-production, whether by large or small doses, so as to realise the maximum of profit,[2] the alert business man acting upon the "principle of Substitution." [3] If it appears that the cost which would be saved by the omission of any branch of production is greater than the yield of the product, of course the branch must be discontinued (unless, indeed, the loss can be converted into gain by a readjustment of factors, or—in a regime of monopoly—by a revision of rates).

But Prime Cost is, I think, usually taken in a somewhat different, or more general, sense, which may be exemplified by omitting in the example just given the "large items of maintenance," and taking account only of operating expenses. In general there is to be excluded in the computation of Prime Cost the cost of some factors of production of the kind above described as varying discontinuously. This computation would, of course, be made more readily than the one above described. It might be of some use, though not of as much use as the more difficult computation. If it appears that the prime cost of the product in the sense proposed is greater than its yield, the production must be unprofitable. But the converse is not true; the prime cost might be less than the yield, and yet the production might not be (in the long run) profitable.

Taken in the sense proposed Prime Cost makes its appearance under conditions which we have seen to be favourable to Increasing Return and Joint Production. Prime Cost, Joint Cost, Decreasing Cost, may often be predicated of the same circumstances. The three classes are related to each other as circles so intersecting as to have a portion of area in common.

freights presuppose the existence of a railway. And, again, prime cost, even though denoting only operating expenses, may be regarded as a kind of special cost, the cost that could be saved *during a short period* (not admitting of complete readjustment) by the discontinuance of the product.

[1] Maurice Clark, *Local Freight Discrimination*, p. 36.

[2] Above, subsection 3 *et passim*.

[3] *Principles of Economics*, ed. 6, p. 355 *et passim sub voce* "Substitution."

For example, suppose that on a train running from the station X to the terminus P, a car (*Anglice* truck) is put on to carry oysters of two descriptions, some grown at X, others brought to X from Y, a place further from P (via X) than X is. The case is a familiar one, being used in a classical treatise to illustrate discrimination. Suppose that the shipment of oysters is attended with some expense (not necessarily the same for the two descriptions of goods), an expense varying with the variation in the amount of oysters shipped, while the expenses incident to running the car—for wear and tear, etc.—remain the same, whatever the load. Then that cost of handling may be regarded as *prime cost.* Also the services conferred by transporting oysters of the two descriptions are joint products. For the transportation of oysters of one description (in quantities amounting to a substantial fraction of a car-load) necessitating the use of an additional car, facilitates the transportation of oysters of the other description (in like amounts). If, for instance, starting from the zero of oysters shipped we add an increment—amounting, say, to some three-eighths of a car-load—of oysters grown at X, the cost of adding an increment—of like amount—of oysters coming from Y is thereby reduced; it becomes, in fact, only the prime cost of the latter increment. It results from the same circumstances that Increasing Return is exemplified by both kinds of transportation.[1] If, as before, we start with a substantial increment of one kind, then the cost of a second increment of the *same* kind will be only the prime cost of that second increment.[2] So if a train is put on to carry milk as well as passengers, the cost of the wear and tear, etc., of the trucks required for the milk, together with the cost of handling the milk, is prime cost. The carriage of the milk and that of the passengers are joint products. Also each is apt to exemplify Increasing Returns.

The reader will observe here as throughout how the attribution

[1] The Joint Production is of species *b* (above, p. 88) as well as species *c*, (p. 89).

[2] See the definition of Increasing Return, subsection 4 (above, p. 66 *seq*). If, reverting to our original notation, for the factor or its cost x_0 in that passage we put 0 corresponding to zero of oysters shipped, we may put for the cost of transporting oysters amounting to $\frac{3}{8}$ of a carload, $l + \frac{3}{8}\pi$, where l is the fixed cost of running a car, π is the cost of handling a full load, and it is assumed for simplicity that the cost of handling any fractional load is simply proportional to the fraction. Then $f(x_1) = \frac{3}{8}$. Similarly, if we put $l + \frac{3}{4}$ for the cost of transporting x_2, we have $f(x_2) = \frac{3}{4}$. Thus

$$\frac{f(x_2) - f(x_0)}{f(x_1) - f(x_0)} = \tfrac{3}{4}/\tfrac{3}{8} = 2 > \frac{l + \frac{3}{4}\pi}{l + \frac{3}{8}\pi};$$

which inequation is the characteristic of Increasing Return.

of Increasing Return or of Joint Cost presupposes some datum as to the magnitude of the doses employed and other circumstances of the case. Our propositions would not have held good for very small doses of transportation, say a basketful of oysters, which would not necessitate an additional car, nor for very large doses, on the scale say of three-quarters of a car-load, the superposition of which would have necessitated putting on a second oyster-car. A passenger is an increment of very different significance with respect to Joint Production and Increasing Return according as he requires a special train, or helps to crowd a public carriage. Likewise the meaning of Prime Cost varies according to the context, point of view, and purpose in hand.

As was observed with reference to Increasing Return and Joint Production, so also with respect to Prime Cost, the discontinuous dose which cannot be renewed during a short period is not necessarily administered at the beginning of the period. For example, the cost of sorting letters which are to go by a certain train may be taken as (part of) the prime cost of postage, the cost of the train not being taken into account. Yet the letters may be sorted before the train is run.

But the order of time is not indifferent in cases where "quasi-rent" makes its appearance in this connection. I leave it to him who first discerned the importance and distinguished the properties of "quasi-rent," to explain the relation of this conception to "prime," and its correlative "supplementary," cost. We are here concerned only with the general principle underlying the distinction between quasi-rent and profit. We have to observe how differently human action is affected by an object as it appears in the future, and when it has become a *fait accompli*. Not even Jupiter, as the ancients would have said, plans about the past. As the general in a campaign or battle acts *pro re natâ*, not strictly adhering to a preconceived plan, so Directors who would not have counselled investing in a railway that, as it has turned out, yields little profit over and above operating expenses, may still be well advised now in operating that unprofitable railway, since a little is better than nothing.

APPENDIX.

On Some Variant Terminologies.

1. *Professor Carver's Terminology.*—Professor Carver, in his important observations upon Increasing Return,[1] appears to have had in view the species which we have distinguished as *third*. His theory is, I think, specially relevant to the phenomenon here described as "relative discontinuity." [2] This phenomenon appears to be the main ground of the distinction which he draws between the two questions: "What is the best *proportion* in which to combine the various factors? What is the best *size* for the whole business unit?" [3] The distinction is not conspicuous on the hypothesis of perfect continuity proper to the method of variation above labelled γ.* The distinction appears particularly applicable to the case of discontinuity above labelled β.*

Discontinuity also may explain the importance attached by Professor Carver to the limit which separates Increasing from Diminishing Return in the *secondary* sense—the point Q in our Fig. 1. The *secondary* sense enters into a certain proposition which, though a mere truism in the simpler cases, becomes significant where there is more than one *maximum*; [4] the proposition, namely, that, for a given or assigned outlay, the total product is greatest when the average product is greatest. The maxim may be illustrated by a problem which has been already noticed. Suppose that in the case cited from Professor Carver the farmer has a limited amount of capital and labour, say 34 days' labour (with team and tools), to apply to plots of land, which for simplicity we may suppose to be rent free. What number of plots will he find it most profitable to cultivate? [5]

[1] *Distribution of Wealth*, ch. ii.

[2] Above, p. 78.

[3] *Op. cit.*, p. 65.

* The method specified above, p. 77. The Greek letters refer to a passage in the Economic Journal omitted in this Collection.

[4] In the technical sense, distinguished from the *greatest possible*.

[5] Prof Landry's criticism of Prof. Carver in the *Quarterly Journal of Economics*, 1909, calls for notice here so far as it impugns an assumption which we have made throughout: namely, that if x is the amount of commodity produced and z the amount of a factor employed in the production, say $x = f(z)$; then x always increases (or at least never decreases) with the increase of z, $f(z + \Delta z) >$ (or $\nless$) $f(z)$ (Economic Journal, Vol. XXI. p. 351, note 1, *et passim*)—an assumption countenanced by leading theorists, such as Auspitz and Lieben. Consider the diagram used by Flux, Economic Journal, Vol. XV. p. 278. [See below, referred to in Section VI., Vol. II. p. 326.] Certainly in a case like that which is adduced below, p. 96, note 2, the increase of the factor (land)

The circumstance that on the two-plot system labour and capital would be employed on each of the plots in smaller amounts than would give the largest product per unit naturally raises the suspicion that this arrangement is not the best. The suspicion proves, indeed, not to be true, as we have seen.[1] But it well might have been true even in a regime of monopoly had the data been different; [2] and would be true in a regime of perfect competition.

The same phenomenon of relative discontinuity appears to justify the distinction which Professor Carver has drawn in a passage [3] of which the substance is as follows :—Let X (acres of land) with Y (units of labour and capital) produce P product. Then (1) if X with αY produce more than αP (α greater than unity), we have a case of "increasing returns." But (2) if αX with αY produce more than αP, we have "increasing economy of large scale production." The distinction between (1) and (2) is, I think, specially important in the case supposed by Professor Carver in the context, where X (the number of acres of land) varies discontinuously (as compared with the variation of Y)—by doses of ten-acre plots. Yet one may doubt whether the cases are so distinct as to deserve quite different names; and, if so, whether the best names have been adopted.

Firstly, the distinction appears to be one of degree or dimension in this respect, that behind X and Y there is often some Z, which, though supposed constant in the above statement, may, under other circumstances, become multiplied by α.[4] Thus the

is attended with a diminution of the produce; say $x = F(z)$, where $F(z + \Delta z) < F(z)$. But this relation "F" is not identical with that which we have designated "f." For we assume that the entrepreneur "applies his outlay to the best of his ability" (*loc. cit.*, p. 357 and p. 568). Accordingly f does not coincide with F beyond the point at which $F(z + \Delta z)$ becomes less than $F(z)$, if the farmer knows the facts designated by the relation $x = F(z)$; if the farmer does not know the facts, f does not coincide at all with F. We presuppose, of course, common sense on the part of the business man—and of the economist who theorises about business.

[1] Above, p. 78.

[2] For instance, suppose that in order to produce any crop at all there is required a preliminary expenditure of *seventeen* (instead of *two*) days' labour. Then other things remaining the same, every figure in the first column is to be increased by the addition of 15. If the number of available days' labour is now 68, a *maximum* of profit would be afforded by laying out on each of *two* plots 34 days' labour, that is, less than the amount which yields the largest product per plot, now 35 days. But the *greatest possible* profit will be obtained by laying out the whole of the 68 doses on *one* plot.

[3] *Op. cit.*, p. 66.

[4] See above, p. 65, as to the difficulty of using a single simple formula in order to label the diversified relations between Return and Cost; relations which present different characters according to the magnitude of the "doses," the length of the "periods," contemplated.

above statement refers to a single farmer. But if there were several farmers, might not an increase of their numbers, resulting in an improved organisation, lead to a more than proportionate increase of product? And must case (2) be then degraded from "increasing economy of large scale production" to mere "increasing returns"? [1]

Secondly, even if different names are to be given to cases (1) and (2), it may be doubted whether the names proposed are the best. For this nomenclature, as Dr. Marshall has remarked, "would deprive us of an old use of the term which is of great importance; and in which the distribution of the resources of production among different uses is supposed to have been made carefully and well, so far as the knowledge and skill of those engaged in the industry will carry." [2] Accordingly, it is tenable, in the cases above distinguished as (1) and (2), that the terms Increasing or Diminishing Return had better be applied to the *second* case; while the phenomenon defined by Professor Carver in the *first* case as Increasing Return had better be described as failure of the proper proportion [3] between the factors.

2. *Proportions of factors*.—The term "proportion" appears especially suitable to the adjustment of some factors, say X, treated as variable, while some other factor, Y, is treated as constant—the case of relative discontinuity above illustrated; but unsuitable in general, apart from this incident, when all the factors are conceived as varying continuously—the type of variation which we have labelled γ.[4] "Proportion" in this latter use seems to mean nothing more than adjustment of factors so as to obtain the greatest net profit; and this idea is much better expressed by the (greatest possible) maximum value of a function of many variables.[5] Accordingly, I see no advantage

[1] Professor Carver himself admits that "the law of the increasing or diminishing economy of large scale production, while sufficiently distinct from that of increasing or diminishing returns to warrant a difference of name, is yet fundamentally very much like it," *op. cit.* p. 91.

[2] *Principles of Economics*, ed. v. p. 320. Dr. Marshall continues: "The older economists applied the law of Diminishing Return in warnings as to the dangers of the growth of a very dense population . . .; and they consistently assumed that the distribution of resources among different uses would be about the best which were at the command of the population in question."

With respect to such distribution of resources between large and small farms I fail to see that anything is gained by Prof. Davenport's novel terms "Law of Advantage and Size" (*Quarterly Journal of Economics*, Vol. XXIII. p. 610).

[3] In accordance with Prof. Carver's use of that term in a passage cited at the beginning of this Appendix.

[4] ECONOMIC JOURNAL, Vol. XXI. p. 367. [See note * to p. 95 above.]

[5] See note 1 above, p. 76. Compare Marshall, *Principles of Economics*, ed. vi. p. 170: "If his business extends he will extend his uses of each requisite

in substituting for " diminishing return " the phrase " disadvantage accruing from any excess or defect in the relative proportions of the factors of production." [1] A similar substitution would be of no avail in the analogous physical problem, to locate the maximum height of a surface. An Alpinist (prevented, suppose, by a fog from seeing beyond his immediate neighbourhood) requires to know whether he is in a cup-shaped cavity—a " convexa vallis " (convex to the plane of the horizon)—or on a dumpling-shaped surface. He requires the conception of " concave " (and its opposite); and nothing would be gained by substituting such a term as the disproportion between the latitude and longitude of any position, meaning at most its remoteness from the summit. Nothing is gained, and something is lost, by using the term " proportion " where the conception of *function* is required. The single symbol "f" conveys more to the instructed mind than all the words that have been written about the Proportions of Factors.

3. *Professor Chapman's Terminology.*—Some of the preceding points may be illustrated by reference to the original paper in which Professor S. T. Chapman has discussed the *Remuneration of Employers* [2] in connection with Increasing or Diminishing Return. Assuming a community to consist of z similar establishments each with one employer and x employés; he considers the question whether, if an additional employer be taken on, the consequent increment to the total product is greater or less than the remuneration of the average entrepreneur. He assumes that the population zx is constant. He assumes also, as I understand, that the play of competition will bring [3] about a determinate value of z and x. (To fix the ideas, we may suppose that the entrepreneur's remuneration is totally unmixed with rent, so

of production in *due* proportion; but not, as has sometimes been said, *proportionately.*"

[1] " Proportions of Factors," by H. J. Davenport, *Quarterly Journal of Economics*, 1909, Vol. XXIII. pp. 594, 596.

[2] ECONOMIC JOURNAL, Vol. XVI.; [examined in Section VI, γ II. p. 331.]—The " Laws of Increasing and Decreasing Return " which are the subject of Prof. Chapman's article in the ECONOMIC JOURNAL, Vol. XVIII. are to be regarded, I think, as *propositions* of which the predicates are terms defined as here (Prof. Chapman professes agreement with our definition, *loc. cit.*, p. 53), and the subjects are terms more *general* than the subjects of the propositions here contemplated. Compare Prof. Chapman's distinction between the " abstract " and " realistic " statement, in his *Outlines of Political Economy* (1911), p. 105 and context.

[3] As to the play of competition in such a case I may refer to my observations on entrepreneurs' profits in ECONOMIC JOURNAL, see Index, " Entrepreneur." It may be as well to remark that the supposition now made in a parenthesis for the sake of illustration is not necessary for the argument.

that it is open to any worker to transform himself into an entrepreneur, the difference of remuneration compensating for the efforts and sacrifice attending the transformation.) Professor Chapman rightly states that the answer to the question put is affirmative or negative according as Increasing or Diminishing Return acts. But the sense in which these terms are to be taken is not, I think, stated with sufficient precision. In my view the only appropriate sense is a certain one of the subordinate varieties which the secondary definition may present, as above shown in the case of plural factors.[1] Professor Chapman's theorem holds good if by Increasing Return [2] it is meant that (az with x) produces more than a times the product of (z with x). But the theorem does not hold good if by Increasing Return [3] it is meant that (az with ax) produces more than a times (z with x).*

The *primary* definition is not germane to the question above stated. It will be required if the question is: What is the value of z for which the total product is a maximum? But we may go some way towards answering that question without being able to ascertain the character of the Return in the primary sense; if we make the probable assumption that the product of a firm always increases (in virtue of intensified organisation) with the increase of the number of firms *ceteris paribus*.[4] For then, as z is increased (from the value determined by competition), the product of the community would continually increase, as far at least as the point at which the entrepreneur's remuneration dwindles to zero.

[1] *Loc. cit.*, p. 76.
[2] As p. 524, par. 1 (*op. cit.*) must, I think, be interpreted.
[3] As p. 526 note, last par., may, I think, be interpreted.
* The proof of these statements will be found in the original.
[4] The assumption is that $\left(\frac{df}{dz}\right)$ is continually positive.